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LETTER TO THE EDITOR

The rotating elliptic billiard and a signature of quantum chaos

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Abstract. We present the quantum mechanical description of an elliptic billiard which rotates with angular velocity perpendicular to the billiard table. The single particle levels in the rotating frame are evaluated by diagonalisation in the elliptic coordinate basis and the dynamic moment of inertia of the many-particle system is studied as a function of the angular velocity. The irregularities in the behaviour of the moment of inertia are associated with the chaotic dynamics of the classical system.

In recent years, much work has been devoted to investigating if quantum systems, that are chaotic in the classical limit, exhibit some peculiar behaviour that can be understood as quantum chaos. In other words: how does the stochastic behaviour of a classical system manifest in the quantum version of it?

Most of this work has been orientated to the study of the energy spectrum of different integrable and non-integrable Hamiltonians [1]. Even when a precise Hamiltonian is unknown, the nuclear many-body problem can also contribute to answer the question. For example, the close agreement between fluctuations of nuclear levels and predictions of the random matrix theory suggests that the nucleus is a chaotic system, at least at excitation energies of several MeV [2]. Very little is known, however, about other macroscopic magnitudes and observables of quantum many-body systems which are expected to illuminate the problem.

In this letter we deal with the problem of non-interacting particles confined to a hard two-dimensional elliptic box (elliptic billiard) which rotates with angular velocity ω perpendicular to the billiard table. We are going to focus our attention on the dynamic moment of inertia of such a many-particle system. As we will see, the non-integrability of the system will manifest through irregularities in the behaviour of this magnitude. Such irregularities have been interpreted as changes in the nuclear structure when the rotational velocity ω of the nucleus increases leading to the phenomenon called 'backbending' [3]. The present model retains the basic features of the mean field descriptions for deformed axially symmetric nuclei such as cranked Hartree–Fock or cranked Nilsson formalisms [4]. We are thus in the position of using it as a tool to investigate this phenomenon from an alternative perspective.

Let us first recall some of the results of the quantum static elliptic billiard ($\omega = 0$) [5, 6]. The problem is well known to be separable in elliptic coordinates. The single-particle wavefunctions can be factorised in the following way

$$\Psi^{(a)}_{E,b}(\xi, \eta) = \mathscr{R}^{(a)}_{E,b}(\xi) S^{(a)}_{E,b}(\eta) \tag{1}$$

where $\mathscr{R}^{(a)}_{E,b}(\xi)$ and $S^{(a)}_{E,b}(\eta)$ are first-kind radial and angular Mathieu functions of the elliptic coordinates ξ and η. Besides the energy E, the eigenfunctions are also characterised by b, the eigenvalue associated with the conservation in classical mechanics of $\boldsymbol{l}_1 \cdot \boldsymbol{l}_2$, where $\boldsymbol{l}_1(\boldsymbol{l}_2)$ is the angular momentum with respect to the focus at $x=+c(-c)$. The index (a) indicates the y-parity of the Mathieu wavefunction. Alternatively, we can label the basis eigenfunctions as $\Psi^{(a)}_{n_\xi,n_\eta}(\xi,\eta)$ where n_ξ and n_η account for the number of nodes in the pseudo-radial variable ξ and in the pseudo-angular variable η, respectively. If n_η is an even number then the functions correspond to x-symmetric, y-symmetric ($|+_x, +_y\rangle$) states if $a=$ even or to x-antisymmetric, y-antisymmetric ($|-_x, -_y\rangle$) states if $a=$ odd. For odd values of n_η we obtain $|-_x, +_y\rangle$ and $|+_x, -_y\rangle$ states. As is well known, the level spectrum of this system exhibits many crossings when it is studied as a function of μ ($\mu = a/b$, a and b being the major and minor axes).

Suppose now that the billiard table rotates. As long as the potential V is time dependent, the same holds for the Lagrangian and the energy. However, working in the rotating frame, the Hamiltonian (which is no longer the energy) is time independent and we can solve the time-independent Schrödinger equation $\hat{h}^{(\omega)}\Psi = \varepsilon^{(\omega)}\Psi$, where we have defined $\hat{h}^{\omega} = \hat{h}^0 - \omega\hat{l}_z$ which is usually called the Routhian. $\hat{h}^0 = -(\hbar^2/2m)\nabla^2$ and $-\omega\hat{l}_z$ account for the Coriolis and centrifugal forces. The last term breaks the time reversal symmetry but commutes with the operator $\hat{P} = \hat{P}_x \cdot \hat{P}_y$, where $\hat{P}_x$ and $\hat{P}_y$ are the x-parity and y-parity operators with eigenvalues π_x and π_y respectively. The matrix that has to be diagonalised can be reduced to two blocks associated with a quantum number $\sigma = \pi_x \cdot \pi_y$ which is conserved (in nuclear physics σ can be associated with the signature of the state). In the coordinate representation the operator $\hat{l}_z$ can be written

$$\hat{l}_z = \frac{-\mathrm{i}\hbar}{2(\cosh^2 u - \cos^2 v)}\left\{\sin 2v\frac{\partial}{\partial u} + \sin 2u\frac{\partial}{\partial v}\right\} \tag{2}$$

where we have used the transformations $\xi = \cosh u$ and $\eta = \cos v$. By using the Bessel and Fourier expansions of the Mathieu functions the matrix elements can be evaluated in the elliptical basis (1). A little algebra allows us to show that $|++\rangle$ states can only be connected with $|--\rangle$ states (i.e. $\langle ++|l_z|++\rangle = \langle --|l_z|--\rangle = 0$) and the $|+-\rangle$ states only connect with the $|-+\rangle$ ones (the x and y subscripts have been dropped).

The diagonalisation procedure was carried out for each σ using a truncated basis which includes the first 100 states and the accuracy of the calculation was verified by diagonalising an extended matrix of 150 basis states. Figure 1 displays the spectrum that has been obtained for $\sigma = 1$ and a deformation parameter $\mu = 2.2$. These are dimensionless energies and we have also defined a dimensionless frequency $\bar{\omega} = \omega m R_0^2/\hbar$. The area $A = \pi R_0^2 = \pi ab$ has been set constant. The non-integrability of the system is reflected in the presence of multiple repulsions (avoided crossings) and the spectrum is similar to that recently obtained in [7] where a circular billiard rotates about a point on its edge.

To study the many-particle system we assume that each single particle level can be occupied by two particles. Defining $\hat{H}^0 = 2\sum_{i=1}^N \hat{h}_i^0$ and $\hat{L}_z = 2\sum_{i=1}^N \hat{l}_{z_i}$ we obtain the total Routhian $\hat{H}^{\omega} = \hat{H}^0 - \omega\hat{L}_z$ with eigenvalues $E_N^{(\omega)} = 2\sum_{i=1}^N \varepsilon_i^{(\omega)}$. Thus, the ground state of a system with $2N$ particles is obtained by filling the N lowest levels. We are interested in the dynamic moment of inertia per nucleon defined as [3]

$$\mathfrak{F} = \frac{(-1)}{2N}\frac{\partial^2 E}{\partial \omega^2}.$$

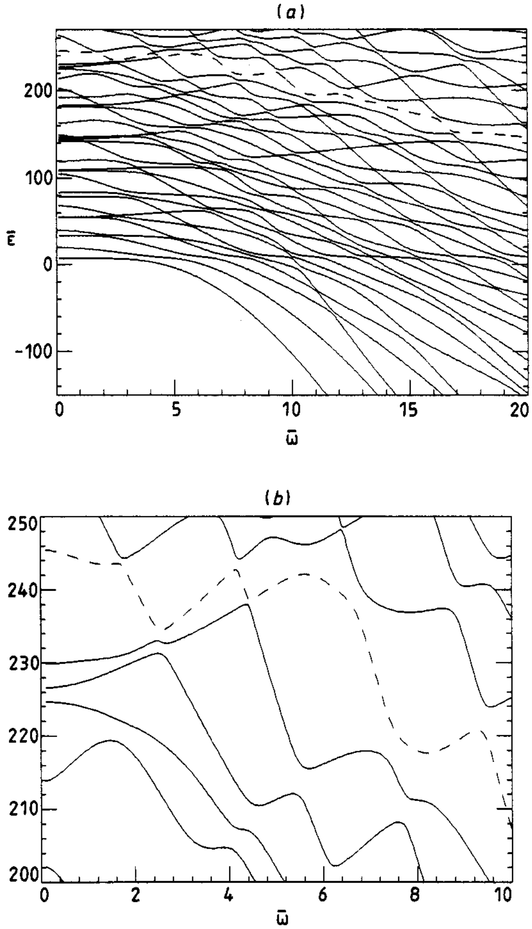

Figure 1. (a) Dimensionless single particle energies $\bar{\varepsilon} = 2mEab/\hbar^2$ for the rotating elliptic billiard as a function of the dimensionless angular velocity $\bar{\omega} = \omega mab/\hbar$. Only the $\sigma = 1$ states are drawn. The 27th level is distinguished by a broken curve. (b) Detail of (a) corresponding to the Fermi level.

We are going to study the dependence of $\mathfrak{F}$ as a function of the angular velocity for fixed deformations μ. The second-order derivative is evaluated numerically through

$$\mathfrak{F} = \frac{(-1)}{2N} \frac{E_N^{(\omega+2\Delta\omega)} + E_N^{(\omega)} - 2E_N^{(\omega+\Delta\omega)}}{(\Delta\omega)^2} \tag{3}$$

with $\Delta\omega = 0.01\ \hbar/mR_0^2$. In order to remove the singularities associated with the true crossings, between states with different signature, we evaluate (3) for each signature. We performed the calculation for the 27 lowest levels with $\sigma = +1$. As displayed in figure 2 the sum of the single particle curvatures is negative leading to positive values of the moment of inertia. As a function of ω, the level curvatures vary slowly except for frequencies in the vicinity of $\omega = \omega_{\text{av.cross.}}$ at which an avoided crossing occurs and an abrupt change takes place. The peaks in the moment of inertia appear when the Fermi level participates as the lowest level of an avoided crossing. In these cases, this state contributes to the sum of (3) with a great negative value which is not cancelled by the positive contribution of the next unfilled level (see figure 1(*b*)).

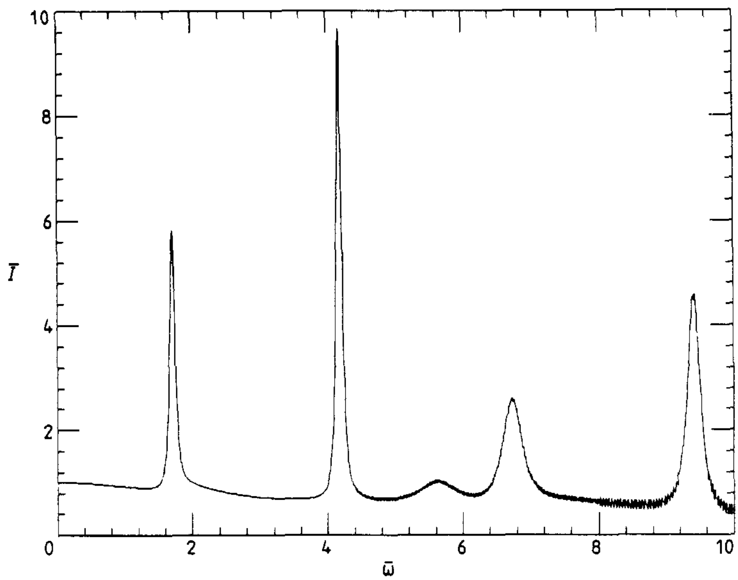

Figure 2. Dimensionless dynamic moment of inertia per nucleon $\bar{I} = \mathfrak{J}/mab$ as a function of $\bar{\omega}$.

Such behaviour of the moment of inertia has frequently been observed in the excitation of high spin states of nuclear rotational bands. It has been called the backbending phenomenon. A difference, however, has to be remarked upon. In nuclear experiments or calculations, the physical observable is the angular momentum of the nucleus, and ω is introduced through a Legendre transformation. For the rotating elliptic billiard that we present here, the angular velocity ω is instead the relevant non-integrability parameter.

It is interesting to note how the large fluctuations of the dynamic moment of inertia—a quantum observable—can be associated with chaotic zones of the classical phase space, at least for slow rotations. As shown in [8], in this low ω condition, the Poincaré sections display essentially the same structure as the integrable non-rotating case [9], except in the vicinity of a broken separatrix, where a strongly chaotic region occurs. This separatrix distinguishes, in the integrable case, the motion that takes place between the two foci ($l_1 \cdot l_2 < 0$) of that outside of the two foci ($l_1 \cdot l_2 > 0$). We have evaluated the expectation values of $l_1 \cdot l_2$ for the Fermi level (the 27th level) as a function of ω in the vicinity of the first avoided crossing ($\omega_{\text{av.cross.}} \approx 1.76$). We found that for ω values smaller than $\omega_{\text{av.cross.}}$, the expectation value is $\langle l_1 \cdot l_2 \rangle < 0$ and changes sign for $\omega > \omega_{\text{av.cross.}}$. The region near the avoided crossing classically corresponds to the separatrix (and chaotic) region. For greater values of ω at which other avoided crossings occur, the structure of the classical dynamics in the phase space becomes completely different and the separatrix region, which corresponds to $l_1 \cdot l_2 \sim 0$ disappears.

The present work shows an effect which is a rotational analogue to the elliptic billiard in a uniform magnetic field studied by Nakamura and Thomas [10] who have related the large fluctuations of the diamagnetic susceptibility with the non-integrability of the system. However, the present system is 'paramagnetic' rather than 'diamagnetic' according to the positive moments of inertia. In fact, the analogy between rotating systems and paramagnetism has been well established by Larmor's theorem [11] and the equivalence between the moment of inertia and the diamagnetic susceptibility is clear in the perturbative expressions of Inglis [12] and Van Vleck [13].

The backbending phenomenon is currently well understood from a nuclear structure standpoint and can qualitatively and quantitatively be described by mean field formalisms. In this paper we stress that behind any such quantum many-body descriptions underlies a chaotic behaviour of the corresponding classical Hamiltonian. In the present example such behaviour is introduced through the Coriolis term which couples the single particle degrees of freedom with the collective motion, simulated by the rotation of the billiard table. An additional important fact is that a macroscopic quantum magnitude is reflecting the chaotic dynamics. We recall that such manifestations of quantum chaos do not show up easily, neither in the spectrum nor in the analysis of the wavefunctions of quantum chaotic systems. In conclusion we suggest that in a realistic three-dimensional case the irregularities in the behaviour of the moment of inertia can be associated with the non-integrability and the chaotic dynamics of the motion of the independent particles inside a rotating nuclear potential well.

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References

[1] Berry M V and Tabor M 1977 *Proc. R. Soc.* A **356** 375
Pechukas P 1983 *Phys. Rev. Lett.* **51** 943; 1985 *Chaotic Behaviour in Quantum Systems* ed G Casati (New York: Plenum)
[2] Bohigas O and Weidenmuller H A 1988 *Adv. Nucl. Part. Sci.* **38** 421
[3] Szymansky Z 1983 *Fast Nuclear Rotation* (Oxford: Clarendon)
[4] Ring P and Shuck P 1980 *The Nuclear Many Body Problem* (New York: Springer)
[5] Ayant Y and Arvieu R 1987 *J. Phys. A: Math. Gen.* **20** 397
Arvieu R and Ayant Y 1987 *J. Phys. A: Math. Gen.* **20** 1115
[6] Traiber A J S, Fendrik A J and Bernath M 1989 *J. Phys. A: Math. Gen.* **22** L365
[7] Siegwart D K 1989 *J. Phys. A: Math. Gen.* **22** 3537
[8] Frisk H and Arvieu R 1988 Transition order-chaos in a rotating billiard *preprint*
[9] Berry M V 1981 *Eur. J. Phys.* **2** 91
[10] Nakamura K and Thomas H 1988 *Phys. Rev. Lett.* **61** 247
[11] Landau L and Lifshitz E 1962 *The Classical Theory of Fields* (New York: Pergamon)
[12] Inglis D R 1954 *Phys. Rev.* **96** 1059
[13] Van Vleck J H 1939 *J. Chem. Phys.* **7** 61